Order relations

A relation C on a set A is a <u>strict total order</u> (or just order relation) if it has the following properties:

1.) If $x, y \in A$ s.t. $x \neq y$, then $x C y$ or $y C x$ (comparability)

2.) For no x does $x C x$ hold (nonreflexivity)

3.) If $x C y$ and $y C z$, then $x C z$ (transitivity)

Note: If $x C y$, then it's not true that $y C x$ ("asymmetry")
Otherwise $x C x$ by 3.), which contradicts 2.)

Ex:
- $<$ is a strict total order on $\mathbb{R}$. (In fact, we use $<$ for order relations)
- $\leq$ is <u>not</u> a strict total order (it's reflexive)
- Define $x C y$ on $\mathbb{R}$ if $x^2 < y^2$ or if $x^2 = y^2$ and $x < y$.
  Check that this is a strict total order.
- The descendant relation among people is not quite a strict total order.
  Satisfies 2.), 3.), but not 1.)

Every <u>strict</u> total order $<$ has an associated total order $\leq$ (not strict) defined $x \leq y \iff x < y$ or $x = y$
(We say a set A along w/ a total order $\leq$ is an <u>ordered set</u>)

Properties:

1.) If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry)

(If $a \leq b$ then $a = b$ or $a < b$. If $a < b$, then $b \not< a$

so $a = b.$)

2.) $a \leq b$ and $b \leq c \implies a \leq c$ (transitivity)

3.) $a \leq b$ or $b \leq a$ (comparability)

In fact, if a relation satisfies these properties, it is a total order — can be used as alternate definition.

Ex:

1.) $\leq$ is the total ordering on $\mathbb{R}$ associated to $<$.

2.) Define $<$ to be the strict total order on words given by alphabetizing (i.e. carrot $\leq$ cucumber)

3.) Let A be a set. The relation $\subseteq$ on $\mathcal{P}(A)$ is not a total order! It's transitive and antisymmetric, but it may be the case that $X \nsubseteq Y$ and $Y \nsubseteq X$ (i.e. not all pairs of subsets are comparable!)

If A is a set and $<$ is an order relation, use the notation (a, b) (resp. $[a,b]$) to denote the set

$$\{x \in A \mid a < x < b\} \quad (\text{resp. } \{x \mid a \leq x \leq b\})$$

Def: A and B sets w/ total orders $\leq_A$ and $\leq_B$, respectively.

A and B have the same <u>order type</u> if there exists a bijection $f : A \to B$ s.t. $a_1 \leq_A a_2 \implies f(a_1) \leq_B f(a_2)$.

Ex: • The subset $(0,1)$ of $\mathbb{R}$ has the same order type as $(1,3)$.

The function $f:(0,1) \longrightarrow (1,3)$ given by

$f(x) = 2x+1$ is the order-preserving bijection.

- $\mathbb{R}$ w/ $\leq$ has the same order type as $\mathbb{R}$ w/ $\geq$

  $f: \mathbb{R} \to \mathbb{R}$ defined $f(x) = -x$ is an order preserving bijection: $x \leq y \Rightarrow -x \geq -y$.

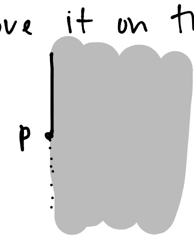
The dictionary order

Let A and B be two sets w/ strict partial orders $<_A$ and $<_B$, respectively. Define an order relation $<$ on $A \times B$ by
$(a_1, b_1) < (a_2, b_2)$ if $(a_1 <_A a_2)$ or $(a_1 = a_2$ and $b_1 <_B b_2)$.

This is called the dictionary order, or the lexicographic order.

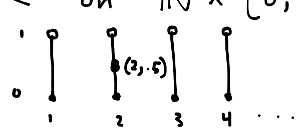
Ex: • Let $<$ be the dictionary order on $\mathbb{R} \times \mathbb{R}$.

Then $p < q$ if q is to the right of the vertical line p is on, or above it on the same vertical line



- Consider the set $[0,1) \subseteq \mathbb{R}$ and $\mathbb{N}$ w/ their usual orders.

  Let $<$ on $\mathbb{N} \times [0,1)$ be the dictionary order



This has the same order type as $\mathbb{R}_{\geq 0}$. The function

$f(a,b) = a+b-1$ is the corresponding order-preserving bijection. (Check: in hw)

Least upper/greatest lower bounds

Let A be a set w/ relation $<$.

Let $A_0 \subseteq A$. b is the largest element (resp. smallest element) of A_0 if $b \in A_0$ and $a \leq b$ (resp. $b \leq a$) $\forall\ a \in A_0$.

Note: Not all sets have a largest/smallest element. e.g. $(0,1]$ has a largest element (1), but no smallest element.

$A_0 \subseteq A$ is bounded above (resp. bounded below) if $\exists\ b \in A$ s.t. $x \leq b$ (resp. $b \leq x$) $\forall\ x \in A_0$. The element b is called an upper bound (resp. lower bound) for A_0.

If the set of all upper bounds for A_0 has a smallest element, it's called the least upper bound or supremum of A_0 and is denoted

$$\sup(A_0).$$

If the set of lower bounds has a largest element, it is called the greatest lower bound, or infimum of A_0 and is denoted

$$\inf(A_0).$$

Ex: In $\mathbb{R}$, $\inf\big((0,1)\big) = \inf\big([0,1]\big) = 0$. (Why are they equal?)

Def: An ordered set A has the *least upper bound property* if every nonempty subset A_0 of A that is bounded above has a supremum. The *greatest lower bound property* is defined analogously.

Ex: • The set $\mathbb{R} \setminus \{0\}$ does *not* have the L.U.B. property:

$(-\infty, 0)$ is bounded above by every element of $(0, \infty)$, but it has no *least* upper bound.

• $\mathbb{R}$ does have the L.U.B. property, as does $\mathbb{Z}$ w/ standard ordering.

Theorem: If an ordered set A has the least upper bound property, then it has the greatest lower bound property.

Proof: Let A be an ordered set w/ the l.u.b. property.

Let $\overset{\neq \emptyset}{A_0} \subseteq A$ be a set that is bounded below. Let

$$B = \{ b \in A \mid b \text{ is a lower bound of } A_0 \}$$

Let $a \in A_0$. Then $b \leq a$ $\forall$ $b \in B$. Thus B is bounded above, so it has a l.u.b. $b_0 = \sup(B)$.

Claim: b_0 is the greatest lower bound for A_0.

On hw: finish proof of Thm by proving the claim.